N-Galilean conformal algebras and quantum theory with higher order time derivatives

K. Andrzejewski*, J. Gonera, P. Kosiński

Department of Theoretical Physics and Computer Science,

University of Łódź,

Pomorska 149/153, 90-236 Łódź, Poland

Abstract

It is shown that centrally extended N-Galilean conformal algebra, with N-odd, is the maximal symmetry algebra of the Schrödinger equation corresponding to the free Lagrangian involving $\frac{N+1}{2}$-th order time derivatives.

It is well known that Schrödinger group is the maximal symmetry group of free classical motion, while its central extension is the maximal symmetry group of the Schrödinger equation of free particle [1]. Recently, Gomis and Kamimura [2] have showed that the free higher-derivative theory

$$\frac{d^{2n}\vec{q}}{dt^{2n}} = 0. \tag{1}$$

defined by the Lagrangian

$$L = \frac{m}{2}\left(\frac{d^n\vec{q}}{dt^n}\right)^2, \tag{2}$$

where $\vec{q}$ is the coordinate in d-dimensional Euclidean space and m is a "mass" parameter of dimension $\mathrm{kg}\cdot\mathrm{s}^{2(n-1)}$ has a symmetry described by N-Galilean

*e-mail: k-andrzejewski@uni.lodz.pl

conformal algebra (N-GCA) with $N = 2n - 1$ (for more information about N-GCA and its relations with higher order time derivatives, see [5]-[10] and references therein). Moreover, they showed that its quantum counterpart, that is the Schrödinger equation

$$i\partial_t \psi = H\psi, \quad \psi = \psi(t, \vec{q}^1, \ldots, \vec{q}^{\frac{N+1}{2}}) \tag{3a}$$

where

$$H = \sum_{j=1}^{\frac{N-1}{2}} \vec{p}_j \vec{q}^{j+1} + \frac{1}{2m} \vec{p}^2_{\frac{N+1}{2}} \tag{3b}$$

is the Ostrogradski Hamiltonian [3] of (2), exhibits centrally extended N-GCA symmetry. In the previous paper [4] the authors showed that N-GCA is the maximal symmetry algebra of the Lagrangian (2). Here, we generalize Niederer's work [1] and show that the centrally extended N-GCA is the maximal symmetry algebra of the Schrödinger equation (3). In order to do this let us recall that the Lie algebra of the maximal Lie group which does not change equation (3) under the change of ψ

$$\begin{aligned} &\psi(t, \vec{q}^1, \ldots, \vec{q}^{\frac{N+1}{2}}) \rightarrow (T_g\psi)(t, \vec{q}^1, \ldots, \vec{q}^{\frac{N+1}{2}}) = \\ &= f_g(g^{-1}(t, \vec{q}^1, \ldots, \vec{q}^{\frac{N+1}{2}}))\psi(g^{-1}(t, \vec{q}^1, \ldots, \vec{q}^{\frac{N+1}{2}})) \end{aligned} \tag{4}$$

consist of the operators X

$$-iX(t, \vec{q}^1, \ldots, \vec{q}^{\frac{N+1}{2}}) = \sum_{j=1}^{\frac{N+1}{2}} \vec{a}^j \vec{\partial}_j + a\partial_t + c \tag{5}$$

satisfying the following equation

$$[P, X] = i\lambda P, \quad P = i\partial_t - H \tag{6}$$

for a certain function $\lambda = \lambda(t, \vec{q}^1, \ldots, \vec{q}^{\frac{N+1}{2}})$. Let us introduce the following notation

$$\vec{q}^j = (q^j_\alpha), \quad \vec{a}^j = (a^j_\alpha), \quad \vec{\partial}_j = (\partial^\alpha_j), \quad etc. \tag{7}$$

where $\alpha = 1, \ldots, d$ and (if the contrary is not stated explicitly) repeated indices i, j, k etc. (α, β, γ etc.) denote summation from 1 to $\frac{N-1}{2}$ (from 1 to d, respectively).

In the case of the Hamiltonian (3b) condition (6) implies the following set of equations for coefficients of the operator X

$$\lambda = \partial_t a + q_\alpha^{k+1}\partial_k^\alpha a \tag{8a}$$

$$\delta_\beta^\alpha = \partial^\alpha_{\frac{N+1}{2}} a_\beta^{\frac{N+1}{2}} + \partial^\beta_{\frac{N+1}{2}} a_\alpha^{\frac{N+1}{2}} \tag{8b}$$

$$0 = i\partial_t c + iq_\alpha^{k+1}\partial_k^\alpha c + \frac{1}{2m}(\partial^\alpha_{\frac{N+1}{2}}\partial^\alpha_{\frac{N+1}{2}})c \tag{8c}$$

$$0 = i\partial_t a_\alpha^{\frac{N+1}{2}} + iq_\beta^{k+1}\partial_k^\beta a_\alpha^{\frac{N+1}{2}} + \frac{1}{2m}(\partial^\beta_{\frac{N+1}{2}}\partial^\beta_{\frac{N+1}{2}})a_\alpha^{\frac{N+1}{2}} + \frac{1}{m}\partial^\alpha_{\frac{N+1}{2}}c \tag{8d}$$

$$0 = \partial^\alpha_{\frac{N+1}{2}} a \tag{8e}$$

$$0 = \partial^\alpha_{\frac{N+1}{2}} a_\beta^j, \quad j = 1, \ldots, \frac{N-1}{2} \tag{8f}$$

$$\lambda q_\beta^{j+1} = \partial_t a_\beta^j + q_\alpha^{k+1}\partial_k^\alpha a_\beta^j - a_\beta^{j+1}, \quad j = 1, \ldots, \frac{N-1}{2} \tag{8g}$$

Our main task is to show that the general solution of the above set of equations gives centrally extended N-GCA. In order to simplify our considerations we assume that $N > 3$. The case $N = 3$ is simpler and can be obtained in the same way.

First, let us note that (8a) and (8e) imply

$$\partial^\beta_{\frac{N+1}{2}}\partial^\alpha_{\frac{N+1}{2}}\lambda = 0 \tag{9}$$

On the other hand differentiating (8g) (for $j = \frac{N-1}{2}$) with respect to $\partial^\alpha_{\frac{N+1}{2}}$ (without summation) and using (8f) and (8b) we obtain

$$\partial^\alpha_{\frac{N-1}{2}} a_\alpha^{\frac{N-1}{2}} = \frac{3}{2}\lambda + q_\alpha^{\frac{N+1}{2}}\partial^\alpha_{\frac{N+1}{2}}\lambda \tag{10}$$

Next, we differentiate the above equation with respect to $\partial^\gamma_{\frac{N+1}{2}}$ and use (9) together with (8f) to obtain

$$\partial^\gamma_{\frac{N+1}{2}}\lambda = 0 \tag{11}$$

Differentiating eq. (8b) with respect to $\partial^\gamma_{\frac{N+1}{2}}$ and combining equations obtained by cyclic permutations of α, β, γ we arrive at

$$\partial^\gamma_{\frac{N+1}{2}}\partial^\alpha_{\frac{N+1}{2}} a_\beta^{\frac{N+1}{2}} = 0 \tag{12}$$

and consequently

$$a_\beta^{\frac{N+1}{2}} = \frac{\lambda}{2} q_\beta^{\frac{N+1}{2}} + U_\beta^\alpha q_\alpha^{\frac{N+1}{2}} + d_\beta \tag{13}$$

where $U = -U^T$ and d_β do not depend on $q_\alpha^{\frac{N+1}{2}}$.

Next, we will show inductively that for each $j = 1, \ldots, \frac{N+1}{2}$ we have

$$\partial_j^\beta a_\alpha^j = \left(\frac{N+2}{2} - j\right) \lambda \delta_\alpha^\beta + U_\alpha^\beta, \tag{14a}$$

$$\partial_j^\beta a = 0, \qquad \text{and if } j > 1 \text{ then } \partial_j^\beta a_\alpha^k = 0 \text{ for } k = 1, \ldots, j-1 \tag{14b}$$

Indeed, due to (8e),(8f) and (13) for $j = \frac{N+1}{2}$ the above assertion holds. Assume that it is true for fixed j. Then, by virtue of (8a) and the induction hypothesis, for index j we have

$$\partial_j^\alpha \partial_j^\beta \lambda = 0 \tag{15}$$

On the other hand, differentiating (8g) for $j-1$ with respect to ∂_j^γ and using the induction hypothesis we arrive at

$$\partial_{j-1}^\gamma a_\alpha^{j-1} = (\frac{N+4}{2} - j) \lambda \delta_\alpha^\gamma + U_\alpha^\gamma + q_\alpha^j \partial_j^\gamma \lambda \tag{16}$$

Putting $\gamma = \alpha$ in (16) and differentiating with respect to ∂_j^β, one gets by (15) and (14b)

$$\partial_j^\beta \lambda = 0 \tag{17}$$

As a result eq. (16) takes the form of (14a) for $j-1$ and, due to (8a), $\partial_{j-1}^\beta a = 0$. Moreover, differentiating (8g) for $k = 1, \ldots, j-2$ with respect to ∂_j^β, by the induction hypothesis and (17) one obtains $\partial_{j-1}^\beta a_\alpha^k = 0$ for $k = 1, \ldots, j-2$ which completes the proof of (14).

Additionally, by virtue of (14b) and (8a) we conclude that a and λ are functions of t only. Differentiating eq. (14a) with respect to ∂_j^β (without summation) and combining with equations obtained by cyclic permutations of α, β, γ we get U_β^α is also only function of t.

Now, differentiating (8g) for $j = 1, \ldots, \frac{N-1}{2}$ with respect to ∂_j^γ (without summation) and using (14a) we obtain a recurrence formula for $\partial_j^\gamma a_{j+1}^\alpha$

$$\partial_t \partial_j^\gamma a_\alpha^j + \underbrace{\partial_{j-1}^\gamma a_\alpha^j}_{0 \text{ for } j=1} = \partial_j^\gamma a_\alpha^{j+1} \tag{18}$$

which, due to (14a), can be explicitly solved and the final result reads

$$\partial_j^\gamma a_{j+1}^\alpha = (-j+N+1)\frac{j\dot{\lambda}}{2}\delta_\alpha^\gamma + j\dot{U}_\alpha^\gamma, \quad j=1,\dots,\frac{N-1}{2} \tag{19}$$

Similarly, differentiating (8g) with respect to ∂_{j-1}^γ we obtain a recurrence formula for $\partial_{j-1}^\gamma a_\alpha^{j+1}$, $j=2,\dots,\frac{N-1}{2}$ which solution is of the form

$$\partial_{j-1}^\gamma a_\alpha^{j+1} = (-2j^2+3j(N+2)-3N-4)\frac{j\ddot{\lambda}}{12}\delta_\alpha^\gamma + \frac{(j-1)j}{2}\ddot{U}_\alpha^\gamma, \tag{20}$$

Let us now study the behaviour of c. Differentiating (8d) with respect to $\partial^\delta_{\frac{N+1}{2}}\partial^\gamma_{\frac{N+1}{2}}$ and using (19) together with (14a) we find that the third order derivative of c with respect to $q_\alpha^{\frac{N+1}{2}}$ is zero, so

$$c = c_1^{\alpha\beta} q_\alpha^{\frac{N+1}{2}} q_\beta^{\frac{N+1}{2}} + c_2^\alpha q_\alpha^{\frac{N+1}{2}} + c_3 \tag{21}$$

where $c_1^{\alpha\beta} = c_1^{\beta\alpha}, c_2^\alpha, c^3$ do not depend on $q_\alpha^{\frac{N+1}{2}}$. Substituting, (21) into (8c) and comparing terms with $q^{\frac{N+1}{2}}$'s one gets the following set of equations

$$0 = \partial^{(\gamma}_{\frac{N-1}{2}} c^1_{\alpha\beta)} \tag{22a}$$

$$0 = q_\gamma^{k'+1}\partial_{k'}^\gamma c_1^{\alpha\beta} + \partial^{(\beta}_{\frac{N-1}{2}} c_2^{\alpha)} + \partial_t c_1^{\alpha\beta} \tag{22b}$$

$$0 = q_\gamma^{k'+1}\partial_{k'}^\gamma c_2^\alpha + \partial^\alpha_{\frac{N-1}{2}} c^3 + \partial_t c_2^\alpha \tag{22c}$$

$$0 = q_\gamma^{k'+1}\partial_{k'}^\gamma c_3 + i\partial_t c_3 + \frac{1}{m} c_1^{\alpha\alpha} \tag{22d}$$

where index with subscript " ' " runs from $1,\dots,\frac{N-3}{2}$ and $(\alpha,\dots)$ is symmetrization over the enclosed indices. On the other hand, differentiating (8d) with respect to $\partial^\gamma_{\frac{N+1}{2}}$ and next substituting (13) for $j=\frac{N+1}{2}$ and (19) for $j=\frac{N-1}{2}$ one gets

$$c_1^{\alpha\beta} = -\frac{mi}{16}(N+1)^2\dot{\lambda}\delta^{\gamma\alpha} \tag{23a}$$

$$\dot{U}_\alpha^\beta = 0 \tag{23b}$$

Consequently c_1's are functions of t only, using this fact, (19),(8d) and (20) for $j=\frac{N-1}{2}$ we obtain

$$\partial^\gamma_{\frac{N-1}{2}} c_\alpha^2 = -\frac{im}{48}\ddot{\lambda}\delta_\alpha^\gamma(N^2-1)(2N+3) \tag{24}$$

Substituting (24) and (23) into (22b), one finds that

$$\ddot{\lambda} = 0 \tag{25}$$

Thus, by virtue of (20), we have

$$\partial^{\gamma}_{j-1} a^{j+1}_{\alpha} = 0, \quad j = 2, \dots, \frac{N-1}{2} \tag{26}$$

A simple consequence of (26) is

$$\partial^{\gamma}_{1} a^{j}_{\alpha} = 0, \quad j = 3, \dots, \frac{N+1}{2} \tag{27}$$

Indeed, differentiating (8g) with respect to ∂^{γ}_{1} we have $(\partial_t + q^{k+1}_{\beta}\partial^{\beta}_{k})(\partial^{\gamma}_{1} a^{j}_{\alpha}) = \partial^{\gamma}_{1} a^{j+1}_{\alpha}$. This together with eq. (26) for $j = 2$ imply (27).

Now, we show that for each $k = 2, \dots, \frac{N-1}{2}$ we have

$$\partial^{\gamma}_{j-k} a^{j}_{\alpha} = 0, \quad j = k+1, \dots, \frac{N+1}{2} \tag{28}$$

Indeed, for $k = 2$ eq. (28) holds due to (26). Assume now (28) is true for $k-1$; we will show that it holds for k. Differentiating (8g) with respect to $\partial^{\gamma}_{j-k+1}$, by the induction hypothesis, one obtains

$$\partial^{\gamma}_{j-k} a^{j}_{\alpha} = \partial^{\gamma}_{j-k+1} a^{j+1}_{\alpha}, \quad j = k+1, \dots, \frac{N+1}{2} \tag{29}$$

but for $j = k+1$, due to (27), we have $\partial^{\gamma}_{1} a^{k+1}_{\alpha} = 0$ which proves (28).

Let us note that (28) implies $\partial^{\gamma}_{k} a_{\alpha}^{\frac{N+1}{2}} = 0$ for $k = 1, \dots, \frac{N-3}{2}$; therefore

$$a_{\alpha}^{\frac{N+1}{2}} = \frac{\lambda}{2} q_{\alpha}^{\frac{N+1}{2}} + \dot{\lambda}\frac{(N+3)(N-1)}{8} q_{\alpha}^{\frac{N-1}{2}} + U^{\beta}_{\alpha} q_{\beta}^{\frac{N+1}{2}} + f_{\alpha}(t) \tag{30}$$

Substituting this in (8d) we find that c^{2}_{α} is a function of t only, more precisely we have

$$c^{2}_{\alpha} = -im\dot{f}_{\alpha} \tag{31}$$

Furthermore, the partial derivatives of c_3 are expressed in terms of c_2:

$$\partial^{\alpha}_{k} c^{3} = (-1)^{\frac{N-2k+1}{2}} \partial_t^{(\frac{N-2k+1}{2})} c^{2}_{\alpha}, \quad k = 1, \dots, \frac{N-1}{2} \tag{32}$$

this can be proved inductively differentiating eq. (22d) with respect to $\partial^\alpha_{\frac{N-1}{2}-l}$ for $l = 0, \ldots, \frac{N-3}{2}$ and using eqs. (22c), (31). Differentiating (22d) with respect to ∂^γ_1 and using eq. (32) one gets

$$\partial_t^{(\frac{N+1}{2})} c^2_\alpha = 0 \tag{33}$$

Thus, by (31), we have

$$c^2_\alpha = \sum_{k=0}^{\frac{N-1}{2}} A^k_\alpha t^k, \quad f_\alpha = \frac{i}{m} \sum_{k=0}^{\frac{N-1}{2}} A^k_\alpha \frac{t^{k+1}}{k+1} + D \tag{34}$$

where A^k_α and D are some constants.

Summarizing, due to (22d) and (23a), (32) we have

$$c_3 = \frac{(N+1)^2}{16} d\lambda - (-1)^{\frac{N-2k'+1}{2}} q^{k'+1}_\alpha \partial_0^{(\frac{N-2k'-1}{2})} c^2_\alpha + C \tag{35}$$

where C is a constant.

Now, it remains to find the dependence a^j_α of t. By virtue of (8g) and equations (14),(19) and (26) we have

$$a^{j+1}_\alpha = \partial_t a^j_\alpha - f^j_\alpha, \quad j = 1, \ldots, \frac{N-1}{2} \tag{36}$$

where $f^j_\alpha = \lambda(j - \frac{N}{2}) q^{j+1}_\alpha + \frac{\dot\lambda}{2}(j - N - 2)(j-1) q^j_\alpha - U^\beta_\alpha q^{j+1}_\beta$ and by definition $f^j_\alpha = 0$ for $j < 1$. Since $\ddot\lambda = 0$, the explicit solution of (36) is of the form

$$a^j_\alpha = \partial_t^{(j-1)} a^1_\alpha - \partial_t f^{j-2}_\alpha - f^{j-1}_\alpha, \quad j = 2, \ldots, \frac{N+1}{2} \tag{37}$$

Comparing eq. (37) for $j = \frac{N+1}{2}$ with eq. (30) we obtain that $f_\alpha = \partial_t^{(\frac{N-1}{2})} a^1_\alpha$ and consequently, by virtue of (14) and (25)

$$a^1_\alpha = \sum_{k=0}^{\frac{N-3}{2}} B^k_\alpha t^k + \frac{N}{2} \lambda q^1_\alpha + U^\beta_\alpha q^1_\beta + \underbrace{\int \ldots \int}_{\frac{N-1}{2}} f_\alpha dt \tag{38}$$

where G^k_α are some constants. Coming back to (37) we arrive at

$$a^j_\alpha = \sum_{l=0}^{\frac{N-1}{2}-j} B^{j+l-1}_\alpha \frac{(j+l-1)!}{l!}t^l + \frac{i}{m}\sum_{k=0}^{\frac{N-1}{2}} A^k_\alpha \frac{k!t^{k-j+\frac{N+3}{2}}}{(k-j+\frac{N+3}{2})!} + \sum_{l=0}^{\frac{N-1}{2}-j} C^l t^l$$
$$+ D\frac{t^{\frac{N+1}{2}-j}}{\frac{N+1}{2}-j} + \frac{\dot{\lambda}}{2}(j-1)(N-j+2)q^{j-1}_\alpha + \frac{\lambda}{2}(N-2j+2)q^j_\alpha + U^\beta_\alpha q^j_\beta \tag{39}$$

Shifting B's by C's or D, after some indices manipulations we find that a^j_α, for $j = 1,\dots,n$ take the form

$$\begin{aligned} a^j_\alpha &= \sum_{l=j-1}^{\frac{N-1}{2}} B^l_\alpha \frac{l!}{(l-j+1)}t^{l-j+1} + \frac{i}{m}\sum_{k=0}^{\frac{N-1}{2}} A^k_\alpha \frac{k!t^{k-j+\frac{N+3}{2}}}{(k-j+\frac{N+3}{2})!} \\ &+ \frac{\dot{\lambda}}{2}(j-1)(N-j+2)q^{j-1}_\alpha + \frac{\lambda}{2}(N-2j+2)q^j_\alpha + U^\beta_\alpha q^j_\beta \end{aligned} \tag{40}$$

Due to eq.(25) $\lambda = 2Et + F$ thus (8a) yields $a = Et^2 + Ft + G$.

Summarizing, we see that all a's and c depend on some constants C, E, F, G, A's, B's and U's. Thus the maximal symmetry algebra of (3) is finite-dimensional and its basis is obtained by selecting the coefficient related to these constants. After troublesome indices manipulations, we find all generators:

iG

$$H = -i\partial_t \tag{41a}$$

iF

$$D = -it\partial_t - i\sum_{j=1}^{\frac{N+1}{2}}(\frac{N}{2} - j + 1)q^j_\alpha \partial^\alpha_j - i\frac{1}{16}(N+1)^2 d \tag{41b}$$

iE

$$\begin{aligned} K = &-it^2\partial_t - i\frac{1}{8}(N+1)^2 dt - i\sum_{i=1}^{\frac{N+1}{2}}\big((j-1)(N-j+2)q^{j-1}_\alpha \partial_j \\ &-it(N-2j+2)q^j_\alpha \partial^\alpha_j\big) - m\frac{(N+1)^2}{8} q_\alpha^{\frac{N+1}{2}} q_\alpha^{\frac{N+1}{2}} \end{aligned} \tag{41c}$$

iB^l_α, for $l = 0, \dots, \frac{N-1}{2}$

$$P^\alpha_l = -il!(\sum_{k=0}^{l} \frac{t^{l-k}}{(l-k)!}\partial^\alpha_{k+1}) \tag{41d}$$

$-\frac{(j-\frac{N+1}{2})!}{mj!}A^\alpha_{j-\frac{N+1}{2}}$, for $j = \frac{N+1}{2}, \dots, N$

$$P^\alpha_j = -ij!\sum_{l=0}^{\frac{N-1}{2}} \frac{t^{j-l}}{(j-l)!}\partial^\alpha_{l+1} - mj!\sum_{k=\frac{N+1}{2}}^{j} q^{N-k+1}_\alpha(-1)^{\frac{N+1}{2}+k}\frac{t^{j-k}}{(j-k)!}, \tag{41e}$$

$\frac{i}{2}U^\beta_\alpha$

$$J^\alpha_\beta = -i(q^j_\beta\partial^\alpha_j - q^j_\alpha\partial^\beta_j) \tag{41f}$$

$\frac{C}{m}$

$$Z = m \tag{41g}$$

The obtained generators agree with the ones from [2] and do satisfy N-GCA commutation rules with central charge $Z = m$. Thus centrally extended N-GCA is in fact the maximal symmetry algebra of the Schödinger equation corresponding to the free theory with higher order time derivatives.

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